Linear Algebra Problem Book.

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Linear Algebra Problem Book. By Paul R. Halmos. The Mathematical Association of America, Washington, 1995, xiii + 336 pp., $39.95.

Reviewed by **Robert Messer**

The preliminary title of my linear algebra text [**1**] was *A Plaid Linear Algebra*. The reviewers thought that was too controversial, and the editor kindly but firmly suggested that I come up with a more descriptive title. Perhaps the reviewers were reading too much into the title. I was merely suggesting that an undergraduate course in linear algebra can be founded on subject matter that is intrinsically interesting, on methodology that is distinctive to mathematics, as well as on applications that are useful to other disciplines. Applied linear algebra is too narrow. An introductory course should weave a variety of approaches into a fabric that demonstrates the advantages of abstract mathematics in addition to the power of its applications.

Stop for a moment and enumerate the key concepts of linear algebra. Are Gaussian elimination, matrix multiplication, and determinants high on your list? Probably not. Yet most textbooks methodically lull students into thinking that linear algebra is a fairly dull, computationally oriented course. Furthermore, the first two or three chapters on these topics do little to prepare students for the whirlwind tour of abstract vector spaces, subspaces, linear independence, and dimension that usually falls shortly after the deadline for withdrawal from the course! In fact, once students associate vectors with coordinates, they have trouble relating to the important examples of spaces of functions; they are completely mystified as to why there is any need to prove such "obvious" identities as $0\mathbf{v} = \mathbf{0}$; and linear independence is simply another situation where Gaussian reduction churns out the correct answer.

Why keep students in the dark? Linear algebra is about vector spaces and linear transformations. These feature attractions unify the entire course. With these tools, students are ready to explore some elegant mathematics and impressive applications.

Plunging into abstract vector spaces at the very beginning of a term requires a bit of courage on the part of the instructor. However, many students love it: ever since high school geometry their mathematics education has seemed like one long partial-fractions problem. Most students will suspend disbelief when presented with a general concept as a labor-saving device (eliminating the need to check long lists of properties every time a new example appears) and as a unifying principle (trying together concepts from algebra, geometry, and calculus). Of course the abstract concepts must be reinforced with concrete examples. But this immediately becomes an opportunity to view Cartesian geometry, matrices, and functions from a new perspective, not merely a rehash of old topics.

College mathematics programs can look forward to reaping the harvest from the innovations of the calculus reform movement. Introductory linear algebra courses must move in a similar direction with an emphasis on concepts rather than computation. The student who understands linearity will be able to figure out what

matrix to use in setting up a linear model. The student who merely manipulates matrices is likely to interchange the rows and columns.

For students going on in mathematics, linear algebra serves as a transition to upper-level mathematics courses. In addition to learning the subject matter of linear algebra itself, these students must be fortified with a degree of mathematical maturity in working with axioms and definitions, basic proof techniques, and mathematical terminology and notation. These issues cannot be left to chance; they must be addressed explicitly to prepare students for courses such as abstract algebra and real analysis.

As a textbook for a linear algebra course, Paul Halmos's *Linear Algebra Problem Book* satisfies these criteria. After a brief chapter of nineteen problems to motivate the algebraic laws of a field, the basic theory of finite-dimensional vector spaces follows in the next thirty-four problems. Then come linear transformations, duality, similarity, canonical forms, inner product spaces, and normality, 164 problems in all. There are occasional systems of equations to solve, but no fanfare for Gauss-Jordan reduction in solving them. Likewise, the determinant function is a theoretical tool, not a practical means of computing matrix inverses or finding eigenvalues.

Halmos is brimming with advice about guessing solutions, forming conjectures, unraveling definitions, and dealing with notation (when to be fussy as well as when to be causal). He repeatedly warns of difficulties and suggests methods of surmounting them. He provides thorough motivation for the definitions and theorems. All this is in the context of sharpening skills at solving mathematical problems.

And what a collection of problems! You will know you are not reading a Schaum's Outline. This is one case when I am thankful the author has provided solutions in the back of the book. In fact, the hints and solutions here provide as much reading as the discussion and statements of the problems. Unfortunately, all but the most talented undergraduates are likely to be overwhelmed by this material in a first course in linear algebra, especially if they are expected to derive much of the theory in solving these problems.

Rather than a text, this book might more reasonably suggest explorations that, guided by the solutions, are accessible to undergraduate students. For example, a problem involving norms and inner products asks, "Can two different inner products yield the same norm?" A student is unlikely to discover the polarization law, but might appreciate the significance of this identity in answering the question.

I look forward to posing to my linear algebra class the problem of showing that every eigenvalue of AB is also an eigenvalue of BA. For nonzero eigenvalues, this boils down to proving that if $I - AB$ is invertible, then $I - BA$ is invertible. Even the hint of speculating about a series representation for $(I - AB)^{-1}$ is likely to sail over the heads of most students. But the verification that $(I - BA)^{-1} = I + B(I - AB)^{-1}A$ is a welcome addition to the standard exercises that reinforce the definition of a multiplicative inverse.

Similarly, it seems far-fetched to expect students to develop the essence of the Gram-Schmidt orthonormalization process in response to the question, "Does every inner product space of dimension n have an orthonormal set of n elements?" (even with the hint "Keep enlarging"). But they might make progress and gain some insights if they were guided in working with the formula for the component of a vector orthogonal to an orthonormal set of vectors.

The conversational style of writing in this book occasionally lapses into annoying chattiness. A definition can be guessable and an answer conjecturable. A corollary

can be unsurprising or minute but enchanting. Within three sentences the zero linear functional has two symbols and goes from most trivial to most important and ends up uninteresting. Questions have answers that vary from "trivially yes" and "easily yes" to "no but yes" and "a strong NO." Thoughts can require a microsecond, a few seconds, or a moment. Here is a sentence that required more than a moment's thought: "If a vector u belongs to both M and $M^{\perp}$, then $(u, u) = 0$ (by the definition of $M^{\perp}$): that implies, of course, that $u = 0$, that is that the subspaces M and $M^{\perp}$ are disjoint."

On the other hand, this casualness spiced up the discussion of spans with the forgivable misstatement that the span of $\{(1, 1, 1), (0, 0, 0)\}$ is the line through the origin that makes an angle of 45° with each of the three coordinate axes. It also permitted a reference to black magic in proving a result related to the Schwarz inequality.

From the author who brought us *Finite-Dimensional Vector Spaces* in 1958, we now have this engaging collection of problems to accompany any text in linear algebra.

REFERENCE

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After Math. Puzzles and Brainteasers. By Ed Barbeau. Wall & Emerson, Toronto & Dayton, 1995, x + 198 pp., $14.95.

The Chicken from Minsk. By Yuri B. Chernyak & Robert M. Rose, illustrated by Joseph Latinsky. Basic Books (HarperCollins), New York, 1995, xii + 191 pp., $10.00.

New Mathematical Diversions, revised edition. By Martin Gardner. Mathematical Association of America, Washington, D.C., 1995, 268 pp., $19.95.

Reviewed by **David Singmaster**

An essay on recreational mathematics should begin with a discussion of what recreational mathematics is, or at least what its author thinks it is. Clearly, recreational mathematics is intended to be mathematics that is fun, but this is not sufficiently restrictive. Most mathematicians enjoy what they do, but it would not fit the ordinary definition of "recreational."

One might propose that recreational mathematics is mathematics done without thought of applications, i.e., mathematics done for its own sake. This is too broad, since it includes all of pure mathematics and excludes several branches of recreational mathematics that were developed in response to actual recreations: probability was developed to solve gambling problems; combinatorial game theory was developed to analyze games; Eulerian circuits and Hamiltonian circuits were developed to study actual circuits. So this is not a definitive criterion, but it certainly describes a large field that contains much of recreational mathematics.